State Space Least Mean Fourth Algorithm

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***Abstract*—Adaptive filters generally employed for estimation purposes require high computational power when it comes to real time estimation. Therefore, in this paper we propose a computationally light yet effective estimation algorithm based on state space model. Our algorithm has been employed successfully in linear and non linear state space model based estimation problems. We investigate few examples to demonstrate the novelty of our algorithm by comparison with few existing algorithms in presence of non Gaussian noise namely uniform noise. More specifically, the state space normalized least mean squares and the Kalman filter has been compared with our algorithm.**

***Index Terms*—SSLMF, SSNLMS, State Space Least Mean Fourth, State Estimation Algorithm**

I. Introduction

Advancements in computational speed, complexity and power efficiency of digital processors have assisted in adaptive filters gaining widespread acceptance and implementations in numerous fields. Adaptive filters provide an upper hand in comparison to conventional filters because of their ability to adapt by self adjusting filter parameters according to the optimization algorithm utilized ([1],[2]). Two of the most widely used adaptive filtering algorithms are the least mean squares (LMS) algorithm ([1],[2],[3]) and the recursive least squares (RLS) algorithm ([1],[2],[3]). The state space (SS) version of these algorithms have been developed and presented by Malik et al. ([4],[5],[6],[7]) and different analyses were presented.

Adaptive filters dealing with SS model of a system yields near true results compared to non model based systems because of the availability of prior knowledge of the states via the system model. Numerous adaptive filtering algorithms can be found in the literature addressing the problem of SS based models for example the very well known Kalman filter (KF) [8], which gives the linear optimal solution by calculating the Minimum Mean Square Error (MMSE). Using observations subject to noise and other disturbances, it optimally estimates the system parameters. In the domain of non linear filtering, we have the extended Kalman filter (EKF) [8], unscented Kalman filter (UKF) [9], cubature Kalman filter (CKF) [10], quadrature Kalman filter (QKF) [11] and many other variants of KF [8]. These non linear filtering algorithms are computationally computationally complex and therefore pose difficulties in real time filtering problems due to high computational requirement. In this paper we propose and investigate the state space least mean fourth (SSLMF) algorithm by defining a cost function $\mathbf{J}[k]$ and minimizing it. We also compare our performance with the existing state space normalized least mean square (SSNLMS) algorithm [4] and KF algorithm [8]. Our proposed algorithm is superior due to the fact that it requires lesser computations per iteration compared to the existing linear and non linear model based algorithms (KF, UKF, EKF, CKF, QKF, etc.). The performance of the algorithm in presence of non Gaussian noise namely uniform noise is investigated due to the fact that least mean fourth (LMF) algorithm performs better in non Gaussian noise environments [12].

The paper is organized as such, Section II of the paper introduces the State-Space model. Then in Section III the overview of the existing SSLNMS is presented following Section IV where the SSLMF algorithm is developed. Section V presents simulation and results of the comparison of and proposed algorithm keeping in view the mean square errors. And finally we conclude the paper in Section VI.

II. State-Space Model

We begin by defining the general state-space model of a linear time varying system.

$$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{B}[k]\mathbf{u}[k] + \mathbf{w}[k], \tag{1a}$$
$$\mathbf{y}[k] = \mathbf{C}[k]\mathbf{x}[k] + \mathbf{D}[k]\mathbf{u}[k] + \mathbf{v}[k] \tag{1b}$$

where $\mathbf{x} \in \Re^n$ are the process states, $\mathbf{y} \in \Re^m$ are the measured outputs such that $m \leq n$. $\mathbf{A}[k]$ is the state transition matrix, $\mathbf{B}[k]$ is the input matrix, $\mathbf{u}[k]$ is the input vector where $\mathbf{u} \in \Re^p$, $\mathbf{w} \in \Re^n$ is the process noise vector and $\mathbf{v} \in \Re^m$ is the measurement noise vector. The matrix $\mathbf{C}[k]$ is the output matrix where $\mathbf{dim}[\mathbf{C}[k]] = m \times n$, $\mathbf{D}[k]$ is the feed through matrix with $\mathbf{dim}[\mathbf{D}[k]] = m \times p$. It is assumed that the above system is observable. A special case is the unforced (autonomous) linear time varying system, represented as

$$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{w}[k], \tag{2a}$$
$$\mathbf{y}[k] = \mathbf{C}[k]\mathbf{x}[k] + \mathbf{v}[k] \tag{2b}$$

The state space representation for a non linear continuous time system is

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{w}), \tag{3a}$$
$$\mathbf{y} = h(\mathbf{x}, \mathbf{u}, \mathbf{v}) \tag{3b}$$

where f and h are non linear functions and the parameters are as defined before.

III. Overview of the existing SSNLMS

Considering the system described by equation (2). A model based adaptive estimation process can be divided into the

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following two steps. Step 1, the time update which is given by

$$\bar{\mathbf{x}}[k] = \mathbf{A}[k-1]\hat{\mathbf{x}}[k-1] \tag{4}$$

Step 2, the measurement update which is given by

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] + \mathbf{K}[k]\varepsilon[k] \tag{5}$$

where

$$\varepsilon[k] = \mathbf{y}[k] - \bar{\mathbf{y}}[k] \tag{6}$$

here $\mathbf{y}[k]$ is as mentioned in (2), $\mathbf{K}$[k] is the gain matrix and

$$\bar{\mathbf{y}}[k] = \mathbf{C}[k]\bar{\mathbf{x}}[k] \tag{7}$$

This is basically the structure employed in all Kalman Filtering techniques ([8],[9],[10],[11], etc.). In the case of SSNLMS [4] the gain matrix $\mathbf{K}[k]$ is derived as follows

$$\mathbf{e}[k] = \mathbf{y}[k] - \hat{\mathbf{y}}[k] = \varepsilon[k] - \mathbf{C}[k]\delta[k] \tag{8}$$

where

$$\delta[k] = \hat{\mathbf{x}}[k] - \bar{\mathbf{x}}[k] \tag{9}$$

Assuming $\mathbf{C}[k]$ is full rank, $\hat{\mathbf{x}}[k]$ is chosen such that $\mathbf{e}[k] = 0$ which implies the following

$$\varepsilon[k] = \mathbf{C}[k]\delta[k] \tag{10}$$

$\delta[k]$ is chosen as the minimum norm solution of the above equation which results in

$$\delta[k] = \mathbf{C}^T[k](\mathbf{C}[k]\mathbf{C}^T[k])^{-1}\varepsilon[k] \tag{11}$$

here

$$\mathbf{K}[k] = \mathbf{C}^T[k](\mathbf{C}[k]\mathbf{C}^T[k])^{-1} \tag{12}$$

The estimator equation from (9) and (11) is hence derived as

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] + \mathbf{C}^T[k](\mathbf{C}[k]\mathbf{C}^T[k])^{-1}\varepsilon[k] \tag{13}$$

This is termed in ([4],[5]) as the SSNLMS algorithm. However, to avoid the non invertible scenario of $(\mathbf{C}[k]\mathbf{C}^T[k])^{-1}$, a small value γ might be added to the term and hence the estimator can be represented as

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] + \mathbf{C}^T[k](\gamma + \mathbf{C}[k]\mathbf{C}^T[k])^{-1}\varepsilon[k] \tag{14}$$

The above estimator may be unstable and there are no known conditions for stability and hence convergence is not guaranteed. Therefore, a matrix $\mathbf{G}$ has been introduced in ([4],[5]) to overcome this problem. In ([4],[5]), it is claimed that the choice of this matrix $\mathbf{G}$ depends on the nature of the problem and one simple approach is to take $\mathbf{G}$ to be all zeros except for the first column to have non zero entries. In our investigation, we will consider the SSNLMS algorithm ([4],[5]) defined as

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] + \mu\mathbf{G}\mathbf{C}^T[k](\gamma + \mathbf{C}[k]\mathbf{C}^T[k])^{-1}\varepsilon[k] \tag{15}$$

μ is the step size parameter for quicker achievement of convergence.

IV. Derivation of SSLMF

Considering the class of adaptive filtering based on (5) which is employed in all KF techniques, this type of update equation can be generalized with the following update rule

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] - \mu\nabla\mathbf{J}[k] \tag{16}$$

where $\mathbf{J}[k]$ is the cost function to be minimized, $\nabla\mathbf{J}[k]$ is the gradient and μ is the step size parameter. To derive the SSLMF, we start by defining the cost function as

$$\mathbf{J}[k] = \mathbf{E}\left[\|\varepsilon[k]\|^4\right] \tag{17}$$

Minimizing the cost function $\mathbf{J}[k]$ with respect to the predicted states $\bar{\mathbf{x}}[k]$ result in

$$\nabla\mathbf{J}[k] = -4\|\varepsilon[k]\|^2\mathbf{C}^T[k]\varepsilon[k] \tag{18}$$

Equation (16) can therefore be written as

$$\hat{\mathbf{x}}[k] = \bar{\mathbf{x}}[k] + \mu\mathbf{G}\|\varepsilon[k]\|^2\mathbf{C}^T[k]\varepsilon[k] \tag{19}$$

which is our general estimator algorithm.Comparing equations (5) and (19) yields

$$\mathbf{K}[k] = \mu\mathbf{G}\|\varepsilon[k]\|^2\mathbf{C}^T[k] \tag{20}$$

Matrix $\mathbf{G}$ was imposed for the condition of controllability ([4],[5]) which is required due to the dynamics of the system where the algorithm is being applied.

V. Simulation Results & Discussion

Simulation results are presented here to validate the performance of the algorithm. We perform the estimation of the state parameters of a noisy sinusoid tracking problem, a real time trajectory tracking problem and the state parameters of a Van der poll oscillator ([4],[13]). These problems were investigated in the presence of uniform noise. The process and observation error covariance matrices were chosen to be similar to the noise covariances. An overview of the root mean square error (RMSE) of the output observations can be referred to in Table I. SSNLMS, KF, and SSLMF were compared. 100 simulations were performed while the estimation algorithms ran simultaneously.

A. *Example 1. Tracking Sinusoids*

In the first example we consider the system reported in ([4],[5]). More specifically we investigate a second order transversal filter with known frequency and unknown phase and amplitude of sinusoids which produces a 4^{th} order system given by

$$\mathbf{A}[k] = \mathbf{diag}\left\{\begin{bmatrix}\cos(\omega_i T) & \sin(\omega_i T)\\ -\sin(\omega_i T) & \cos(\omega_i T)\end{bmatrix}\right\},\ i = 1,2 \tag{21a}$$

$$\mathbf{B}[k] = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T \tag{21b}$$

$$\mathbf{C}[k] = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix} \tag{21c}$$

The ω_i's for this system are known and kept constant. For the purpose of our study, we have set the values of the frequencies ω_1, ω_2 to 0.5 and 0.25 respectively. The sampling time is considered as $T = 0.1s$ and μ for SSNLMS was taken as 0.05 whereas, μ for SSLMF was chosen as 300. The observation is subject to uniform noise of covariance $\sigma^2 = 0.01^2$. The process noise was considered to be of covariance $\sigma^2 = 0.001^2$. The actual initial system states were considered to be $\mathbf{x}[0] = [0.1\ 0.1\ 0.1\ 0.1]^T$, and the initial estimate for tracking were chosen to be $\hat{\mathbf{x}}[0] = [0.15\ 0.2\ 0.05\ 0.16]^T$. The plot of observation and the mean square observation error are presented in Figure 1 and Figure 2 respectively. It is observed from Figure 1 that SSLMF performs better in terms of observation. A closer look at Figure 2 reveals that SSLMF maintains a lower level of observation error, while KF has the lowest. However, KF is computationally complex and thus state space least mean algorithms have an advantage due to their light computational requirement. Table I shows that the root mean square observation error for SSLMF is lower than that of SSNLMS and hence performed better than SSNLMS.

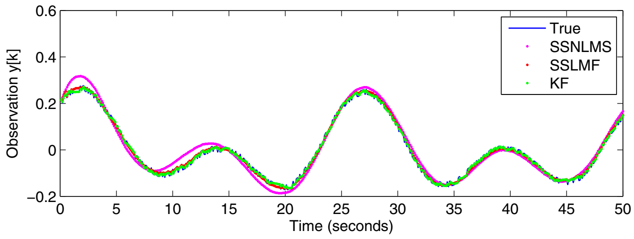

Fig. 1: Observation y[k].

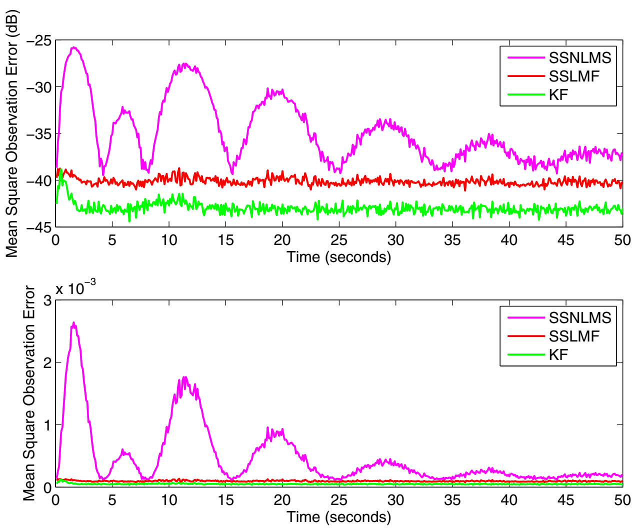

Fig. 2: Mean Square Observation Error y[k]

B. *Example 2. Real time tracking*

Considering the example of real time tracking ([8],[5]), the state space representation is as follows

$$\mathbf{x}[k+1] = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k] \tag{22a}$$

$$y[k] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}[k] \tag{22b}$$

The sampling time T is taken as 0.01s. The initial true state was considered as $\mathbf{x}[0] = [0.4\ 0.3\ 0.4]^T$ and the initial state for the estimation algorithms were chosen as $\hat{\mathbf{x}}[0] = [0.2\ 0.1\ 0.4]^T$. The observation was subject to uniform noise of covariance $\sigma^2 = 0.1^2$ and the process noise was considered to be of covariance $\sigma^2 = 0.01^2$. Figure 3 presents the observation plot while Figure 4 presents the mean square observation error. It was observed that both algorithms effectively estimate the states. However, SSLMF converges fast than SSNLMS which can be observed in 4. Referring to I, it can be confirmed that SSLMF performed better than SSNLMS.

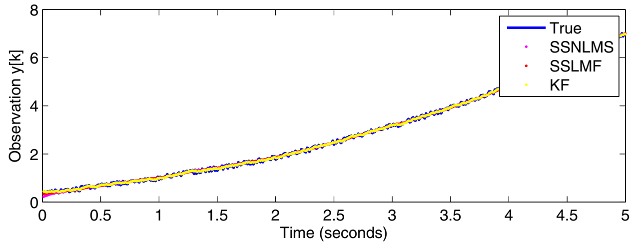

Fig. 3: Observation y[k]

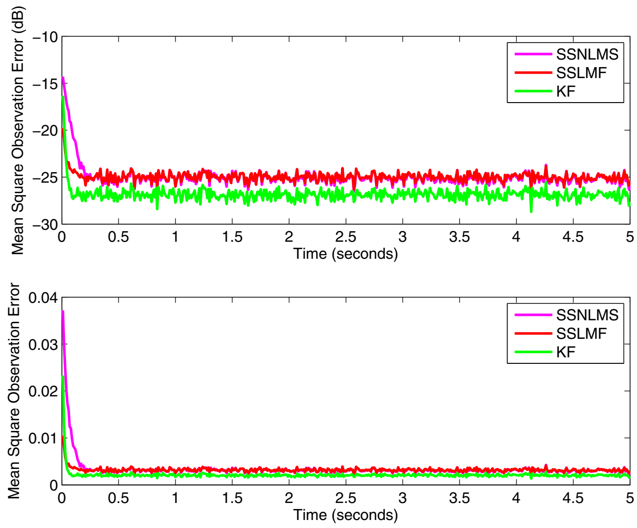

Fig. 4: Mean Square Observation Error y[k]

C. *Example 3. Van der poll oscillator*

In this example we consider the commonly used Van der poll oscillator which is a highly nonlinear system exhibiting both stable and unstable limit cycles [14]. We consider the case of stable limit cycle therefore, as time proceeds the system states converge to zero. The system is represented by the following differential equations ([13],[14])

$$\dot{x}_1 = -x_2 \tag{23a}$$

$$\dot{x}_2 = x_1 - \alpha(1 - {x_1}^2)x_2 \tag{23b}$$

here $\alpha = 0.2$ and the system has been discretized with a sampling time of 0.01s. The state space representation is as follows

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & -\alpha(1 - x_1(t)^2) \end{bmatrix} \mathbf{x}(t) \tag{24a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) \tag{24b}$$

μ for the SSNLMS was chosen as 0.1 and μ for SSLMF as 0.7. The observation was subject to uniform noise of covariance $\sigma^2 = 0.01^2$ and the process noise was considered to be of covariance $\sigma^2 = 0.001^2$. The system was simulated with initial system states to be $\mathbf{x}[0] = [1.4\ 0]^T$, and the initial estimate for the algorithms were chosen to be $\hat{\mathbf{x}}[0] = [1.3\ 0.2]^T$. The observations and the mean square observation error is presented in Figure 5 and Figure 6 respectively. It can be observed from I that the observation error remains lower for SSLMF in comparison to SSNLMS. Moreover, SSLMF converges faster compared to SSNLMS.

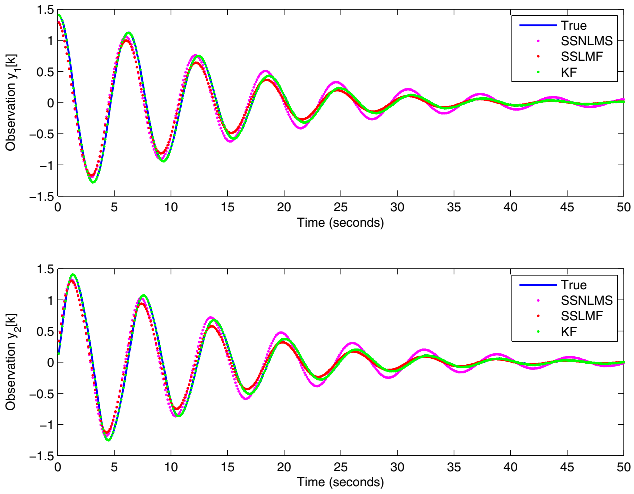

Fig. 5: Observation $y_1[k]$ and $y_2[k]$.

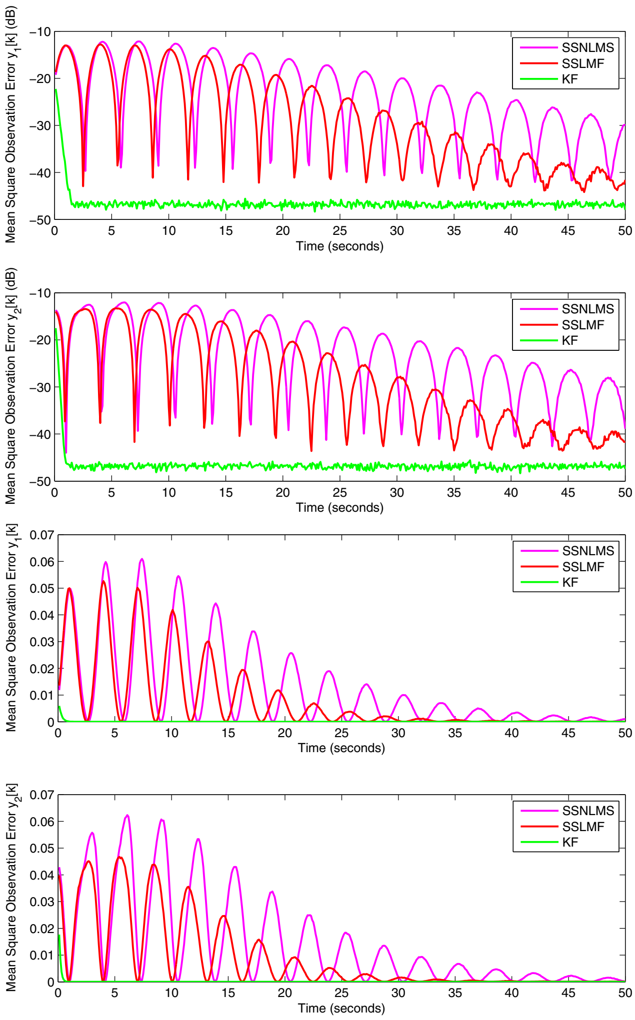

Fig. 6: Mean Square Observation Error for $y_1[k]$ and $y_2[k]$.

It is clear from the study that SSLMF indeed is a novel technique for estimation. Moreover, it is computationally very light when compared to other model based techniques like KF, EKF, UKF, etc. Our study reveals that SSLMF can be effectively utilized in non Gaussian noise environments.

TABLE I: Root Mean Square Error

		Root Mean Square Error		
		SSNLMS	SSLMF	KF
Example 1	Observation y	0.0215	0.0099	0.0071
Example 2	Observation y	0.0587	0.0566	0.0462
Example 3	Observation y_1	0.1108	0.0881	0.0073
	Observation y_2	0.1174	0.0911	0.0093

VI. Conclusion

In this paper the novel SSLMF algorithm has been proposed and investigated. The proposed algorithm is efficient and computationally light compared to existing model based linear and non linear estimation techniques (KF, EKF, UKF, etc.). Three different examples were investigated where SSLMF performed better than the SSNLMS and showed acceptable performance in comparison to KF keeping in mind the fact that KF is highly intensive in computation. For further studies SSLMF for Gussian noise environments can be proposed and investigated along with time varying step size parameters or hybrid algorithms for yet better results.

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